Theory of the Maxwell pressure tensor and the tension in a water bridge

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A water bridge refers to an experimental "flexible cable" made up of pure de-ionized water, which can hang across two supports maintained with a sufficiently large voltage difference. The resulting electric fields within the de-ionized water flexible cable maintain a tension that sustains the water against the downward force of gravity. A detailed calculation of the water bridge tension will be provided in terms of the Maxwell pressure tensor in a dielectric fluid medium. General properties of the dielectric liquid pressure tensor are discussed along with unusual features of dielectric fluid Bernoulli flows in an electric field. The "frictionless" Bernoulli flow is closely analogous to that of a superfluid.

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I. INTRODUCTION

Recent and older observations [1–3] have been made of water bridges stretched across supports that are maintained at large voltage differences. A water bridge is a "flexible cable" made up of pure de-ionized water that has an electric field $\mathbf{E}$ in virtue of an applied voltage across the supports at the ends of the fluid cable. Previously to this work, it was not fully understood what forces hold up the water bridge against the force of gravity. It will here be shown that the forces responsible for holding up the water bridge follow from the Maxwell electric field pressure tensor in dielectric polar fluids. In particular, the water bridge viewed as a flexible cable has an electric field induced tension sufficiently large so as to explain the water bridge support. The need for de-ionized water in the experiment is evidently due prohibiting conductivity effects from masking the insulating dielectric effects. In discussing the general theory of the pressure tensor for isotropic dielectric polar liquids, such as water, hydrostatics, and adiabatic hydrodynamics, shall also be explored.

In Sec. II the thermodynamic laws applied to a fluid polar dielectric are described in detail. It is shown that in the presence of an electric field, there are two different thermodynamic pressures, P and $\widetilde{P}$. These turn out to be eigenvalues of the full pressure tensor as discussed in Sec. III. If an infinitesimal surface area $\delta A_{\perp}$ has a normal perpendicular to the electric field lines, then the pressure force is $P\delta A_{\perp}$. If an infinitesimal surface area $\delta A_{\parallel}$ has a normal parallel to the electric field lines, then the pressure force is $\widetilde{P}\delta A_{\parallel}$. It is shown that these results completely characterize the pressure tensor.

In Sec. IV, we compute the tension in a fluid dielectric cylinder. Although there has been considerable previous work on this subject [4–8] and related subjects [9], the basic physics of that calculation is simply explained. Consider a simple cylinder of length L and cross sectional area $A=\pi R^2$. Suppose that a spatially uniform electric field $\mathbf{E}$ exists in a direction parallel to the cylinder axis. Since the tangential component of the electric field is continuous, the electric field inside the cylinder is the same as the electric field outside the cylinder. On the other hand, the displacement field $\mathbf{D}=\varepsilon\mathbf{E}$ is discontinuous at the end points of the cylinder; i.e., the ends of the cylinder have a charge density of $\pm\sigma$ wherein

$$4\pi\sigma=\Delta D=(\varepsilon-1)E=4\pi\frac{Q}{A}. \tag{1}$$

The end of the cylinder that is at the tail of the electric field vector has charge $+Q$, while the end of the cylinder that is at the arrow of the electric field vector has charge $-Q$. The tension in the cylinder is then evidently given by $\tau=QE$ that in virtue of Eq. (1) reads

$$\frac{\tau}{A}=\left[\frac{\varepsilon-1}{4\pi}\right]E^2. \tag{2}$$

In what follows, the stress [Eq. (2)] will be rigorously derived from the Maxwell pressure tensor within the cylinder. The ratio of the tension to cylinder weight, $Mg=\rho ALg$, obeys

$$\frac{\tau}{Mg}=\left[\frac{(\varepsilon-1)E^2}{4\pi\rho gL}\right], \tag{3}$$

wherein ρ and L represent the mass density and the length, respectively, of the cylinder. Although we have employed free end boundary conditions to the cylinder, the tension τ is a local stress quantity independent of global boundary conditions. In Sec. V, we exhibit a plot of the hanging water bridge flexible cable for experimental values of the parameters in Eq. (3) and find the agreement between theory and experiment to be satisfactory.

In Sec. VI, the hydrostatics of the dielectric fluid in an electric field is explored. The crucial quantity of interest is the mean isothermal polarization per molecule that obeys

$$\mathbf{p} = \alpha_T \mathbf{E} \Rightarrow 4\pi\alpha_T = m\left(\frac{\partial \varepsilon}{\partial \rho}\right)_{T,\mathbf{E}}, \tag{4}$$

wherein m is the mass of a single molecule and α_T is the isothermal polarizability. For the water liquid and vapor phases, the molecular polarizabilities obey $\alpha_T^{gas} \ll \alpha_T^{liquid}$. Fluid dielectric films in strong electric fields adsorbed on insulating walls tend to swell to a large thickness. In Sec. VII, the theory of dielectric polar fluid Bernoulli flows in strong electric fields is discussed. Together with the effect of film thickening, it turns out that the film is capable of crawling up an insulating container wall against the force of gravity and flow as in a siphon over the top of the wall and down on the other side. In Sec. VIII, the role of Bernoulli flows in forming the water bridges will be discussed. Analogies between dielectric polar liquid Bernoulli flow siphon oscillations and superfluid U-tube oscillations will be explored. There is the ability of both water films in high electric fields and superfluid films to climb walls against gravitational forces and even to pass over the tops of these walls over to the other side.

II. THERMODYNAMIC ARGUMENTS

Let $\tilde{f}(\rho, T, \mathbf{D})$ represent the Helmholtz free energy per unit volume for a dielectric fluid of mass density ρ, temperature T, and Maxwell displacement field $\mathbf{D}$,

$$d\tilde{f} = -sdT + \zeta d\rho + \frac{1}{4\pi}\mathbf{E}\cdot d\mathbf{D}, \tag{5}$$

wherein s is the entropy per unit volume, ζ is the chemical potential per unit mass, and $\mathbf{E}$ is the electric field. The thermodynamic pressure that follows from Eq. (5) is as follows:

$$\tilde{P} = \zeta\rho - \tilde{f},$$

$$d\tilde{P} = sdT + \rho d\zeta - \frac{1}{4\pi}\mathbf{E}\cdot d\mathbf{D}. \tag{6}$$

On the other hand, one may employ the free energy,

$$f = \tilde{f} - \frac{1}{4\pi}\mathbf{E}\cdot\mathbf{D},$$

$$df = -sdT + \zeta d\rho - \frac{1}{4\pi}\mathbf{D}\cdot d\mathbf{E}, \tag{7}$$

yielding the pressure

$$P = \zeta\rho - f,$$

$$dP = sdT + \rho d\zeta + \frac{1}{4\pi}\mathbf{D}\cdot d\mathbf{E}. \tag{8}$$

The two different pressures obey

$$P = \tilde{P} + \frac{1}{4\pi}\mathbf{E}\cdot\mathbf{D}. \tag{9}$$

For a dielectric fluid in an electric field, isotropy dictates that $\mathbf{D}$ be parallel to $\mathbf{E}$ even if the detailed equations of state are nonlinear, i.e., isotropy yields a free energy of the form

$$f(\rho, T, \mathbf{E}) \equiv f(\rho, T, E^2), \tag{10}$$

so that Eqs. (7) and (10) imply

$$\mathbf{D} = \varepsilon\mathbf{E}, \tag{11}$$

wherein $\varepsilon(\rho, T, E^2) = -8\pi\partial f(\rho, T, E^2)/\partial(E^2)$. Thus, the two possible fluid pressures in Eq. (9) found from the principle of virtual work obey

$$\tilde{P} = P - \frac{\varepsilon}{4\pi}E^2. \tag{12}$$

It may, at first glance, appear strange that there are two physically different thermodynamic pressures in a dielectric fluid subject to an electric field. However, the situation may be clarified when it is realized that due to the electric field, the pressure is in reality a *tensor*.

III. PRESSURE TENSOR

In order to compute the Maxwell pressure tensor in a fluid dielectric, imagine that the fluid undergoes a strain wherein a fluid particle at point $\mathbf{r}$ is sent to the new point $\mathbf{r}'$. Such a transformation induces a strained length scale $ds^2 = d\mathbf{r}'\cdot d\mathbf{r}'$ described by a metric tensor

$$ds^2 = g_{ij}(\mathbf{r})dr^i dr^j. \tag{13}$$

The pressure tensor P_{ij} is then described in terms of the free energy change due to a metric strain

$$\delta F = \frac{1}{2}\int P_{ij}\delta g^{ij} dV. \tag{14}$$

The free energy variation is described by

$$\delta F = \delta\int f dV = \int \delta f dV + \int f\delta dV. \tag{15}$$

In detail, the volume element of the strained fluid is determined by $g(\mathbf{r}) = \det[g_{ij}(\mathbf{r})]$ via

$$dV = \sqrt{g(\mathbf{r})}d^3\mathbf{r} \Rightarrow \delta dV = -\tfrac{1}{2}g_{ij}\delta g^{ij} dV. \tag{16}$$

Volume variational (16) together with mass conservation, in turn, implies a mass density variation

$$\delta\rho = \tfrac{1}{2}g_{ij}\delta g^{ij}\rho. \tag{17}$$

Furthermore, the change in the magnitude of the electric field is

$$\delta E^2 = \delta g^{ij} E_i E_j. \tag{18}$$

We then employ

$$\delta f = \left(\frac{\partial f}{\partial \rho}\right)_{T,E^2}\delta\rho + \left(\frac{\partial f}{\partial E^2}\right)_{T,\rho}\delta E^2,$$

$$\delta f = \zeta\delta\rho - \frac{\epsilon}{8\pi}\delta E^2. \tag{19}$$

In virtue of Eqs. (15)–(19),

$$\delta F = \frac{1}{2}\int \delta g^{ij}\left[g_{ij}(\zeta\rho - f) - \frac{\varepsilon}{4\pi}E_iE_j\right]dV. \tag{20}$$

From Eqs. (8), (14), and (20) we have the final form [10,11] for the pressure tensor

$$P_{ij} = Pg_{ij} - \frac{\varepsilon}{4\pi}E_iE_j. \tag{21}$$

Alternatively, the pressure tensor is given by

$$P_{ij} = P\delta_{ij} - \frac{\varepsilon E^2}{4\pi}n_in_j \quad \text{with} \quad \mathbf{n} = \frac{\mathbf{E}}{E},$$

$$P_{ij} = P\delta_{ij} + (\tilde{P} - P)n_in_j, \tag{22}$$

wherein Eq. (12) has been invoked.

It is now clear as to why there are two thermodynamic pressures, P and $\tilde{P}$, in Sec. II. For an infinitesimal surface area $\delta A_\perp$ whose normal is perpendicular to the electric field lines, the pressure force is $P\delta A_\perp$. For an infinitesimal surface area $\delta A_\parallel$ whose normal is parallel to the electric field lines, the pressure force is $\tilde{P}\delta A_\parallel$. These results describe completely when to use the pressure P and when to use the pressure $\tilde{P}$.

IV. TENSION IN A FLUID CYLINDER

For a fluid dielectric cylinder of length L and cross sectional area A in a uniform electric field parallel to the axis, let us consider the work done in changing the volume $V=LA$ via

$$dV = AdL + LdA. \tag{23}$$

Employing the pressure tensor [Eq. (22)], one finds that both pressures $\tilde{P}$ and P are required,

$$dW = -\tilde{P}AdL - PLdA. \tag{24}$$

When a fluid is stretched at constant volume, $dV=AdL+LdA=0$,

$$-LdA = AdL \Rightarrow dW = (P - \tilde{P})AdL = \tilde{\tau}dL. \tag{25}$$

The effective tension is thereby

$$\tilde{\tau} = (P - \tilde{P})A = \frac{\varepsilon E^2 A}{4\pi}, \tag{26}$$

wherein Eq. (12) has been invoked. Subtracting the tension that would be present for an electric field in the vacuum, $\tau = \tilde{\tau} - \tau_{vac}$, yields our final result for the tension,

$$\tau = \frac{(\varepsilon - 1)E^2 A}{4\pi}, \tag{27}$$

in agreement with Eq. (2) of Sec. I.

V. HANGING FLEXIBLE CABLE

The water bridge consists of a flexible fluid cable that can be suspended by its end points. As reviewed in the Appendix, the bridge is catenary shaped with mass Mg and tension τ and is fixed at end points with the same vertical height separated by a length L. The hanging flexible equation is given by

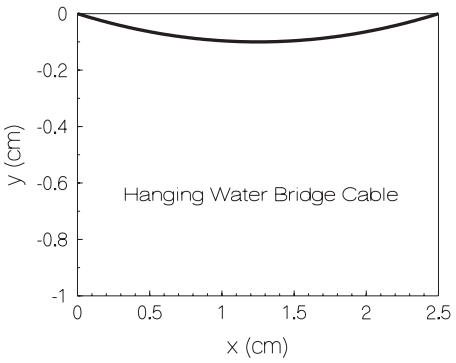

FIG. 1. Employing the experimental example [1,2] wherein the supports are separated by 2.5 cm and with $\rho=1$ gm/cm^3, $g=980$ cm/s^2, $\varepsilon=80$, and $E=10$ kV/cm; we plot the hanging water bridge flexible cable. The maximum hanging dip at the center of the water bridge is $h=0.1$ cm. The agreement between experiment and theory [Eqs. (3) and (29)] is satisfactory.

$$y(x) = a\left\{\cosh\left(\frac{2x - L}{2a}\right) - \cosh\left(\frac{L}{2a}\right)\right\},$$

$$\tan\theta \equiv y'(L) = -y'(0) = \sinh\left(\frac{L}{2a}\right),$$

$$a = \frac{L}{\ln[(1 + \sin\theta)/(1 - \sin\theta)]},$$

$$h = -y_{\min} = -y(x = L/2) = a(\sec\theta - 1). \tag{28}$$

There is a slight sag h in the cable as befits the equilibrium of the total gravitational force Mg downward and the total Maxwell tension force $2\tau\sin\theta$ upward, wherein θ is the angle between the cable tangent at the support and the horizontal,

$$Mg = 2\tau\sin\theta \Rightarrow \sin\theta = \frac{2\pi\rho g L_s}{(\varepsilon - 1)E^2} = \frac{E_0^2}{E^2}, \tag{29}$$

wherein Eq. (3) has been invoked. Equation (29) allows for the theoretical computation of θ in terms of experimental dielectric constants, mass densities, water bridge lengths, and electric fields. With regard to the electric field scale E_0, one finds

$$E_0 = \sqrt{\frac{2\pi\rho g L_s}{\varepsilon - 1}} \approx 2.63\frac{\text{kV}}{\text{cm}}\sqrt{\frac{L_s}{\text{cm}}}. \tag{30}$$

The agreement between theory and experiment in predicting the slight hang of the water bridge as a catenary flexible cable, as in the above (Fig. 1), is satisfactory.

VI. HYDROSTATICS IN AN ELECTRIC FIELD

The force density on the fluid as described by the pressure tensor [Eq. (22)] is given by

$$f_i = -\partial_j P_{ji} = -\partial_i P + \partial_j\left(\frac{\varepsilon E_j E_i}{4\pi}\right). \tag{31}$$

Within the bulk liquid $\mathrm{div}\,\mathbf{D}=\mathrm{div}(\varepsilon\mathbf{E})=0$ so that Eq. (31) thereby reads

$$\mathbf{f} = -\,\mathrm{grad}\,P + \frac{\varepsilon}{4\pi}(\mathbf{E}\cdot\mathrm{grad})\mathbf{E}. \tag{32}$$

On the other hand, from Gibbs-Duhem (8) under equilibrium isothermal conditions $dT=0$,

$$\mathrm{grad}\,P = \rho\,\mathrm{grad}\,\zeta + \frac{\varepsilon}{4\pi}(\mathbf{E}\cdot\mathrm{grad})\mathbf{E}, \tag{33}$$

so that the force per unit volume can be computed from the chemical potential per unit mass: i.e.,

$$\mathbf{f} = \rho\,\mathrm{grad}\,\zeta. \tag{34}$$

Under a Newtonian gravitational field

$$\mathbf{g} = -\,\mathrm{grad}\,\Phi, \tag{35}$$

one finds the total force density equilibrium condition

$$\mathbf{f} + \rho\mathbf{g} = -\,\mathrm{grad}(\zeta + \Phi) = 0, \tag{36}$$

yielding the uniform chemical potential condition

$$\mu = m[\zeta(\rho,T,E^2) + \Phi] = \mathrm{const.} \tag{37}$$

To compute the electric field dependence of the chemical potential per unit mass, one may apply a Maxwell relation to the thermodynamic [Eq. (7)] that reads

$$\left(\frac{\partial\zeta}{\partial\mathbf{E}}\right)_{T,\rho} = -\frac{1}{4\pi}\left(\frac{\partial\mathbf{D}}{\partial\rho}\right)_{T,\mathbf{E}} = -\frac{\alpha_T}{m}\mathbf{E}, \tag{38}$$

wherein the mean molecular dipole moment $\mathbf{p}=\alpha_T\mathbf{E}$ defines the polarizability α_T as in Eq. (4). Integrating Eq. (38) completes the calculation of the chemical potential,

$$\zeta(\rho,T,\mathbf{E}) = \zeta_0(\rho,T) - \frac{1}{2m}\int_0^{E^2}\alpha_T[(\rho,T,F^2)]d(F^2),$$

$$\zeta(\rho,T,\mathbf{E}) = \zeta_0(\rho,T) - \frac{\alpha_T(\rho,T,E^2=0)}{2m}E^2 + \cdots. \tag{39}$$

The central equation [Eq. (39)] of this section implies that the chemical potential is lowered when strong electric fields are applied. We note in passing that a finite (albeit small) conductivity will heat the fluid at a rate per unit volume of $\sigma|\mathbf{E}|^2$. Heating rates may then be substantial in high electric field regions. These may, in turn, be photographed by the resulting infrared radiation.

It follows from Eq. (39) that the application of an electric field lowers the chemical potential of films of water adsorbed on insulating substrates such as glass that is often employed in physical chemistry experiments. When the chemical potential of a liquid film is lowered, the water film thickness increases; e.g., in the presence of electric fields, water in a glass beaker will have a film that appears to climb higher up the walls than would be possible in the zero electric field case. The fabrication of a water bridge begins by applying a potential difference across the water contained in two neighboring glass beakers that just touch each other. One expects and finds experimentally a thickening film layer all around the points at which the water horizontal surfaces meet the two beaker walls. The water climbs the walls of both beakers and near the touch point of the beakers splashes over the top. When the hydrostatic calm after the splash begins, a bridge is formed at the point wherein the beakers just touch. A longer water bridge is formed after slowly separating the beakers. Let us now turn to the hydrodynamic features of the polar liquid flows in a strong electric field.

VII. BERNOULLI FLOWS IN STRONG FIELDS

While there has been interesting previous work [12–14] on strong field Navier-Stokes flows, we here discuss strong field Bernoulli flows since they appear to be more relevant to the water bridge. In particular, when the strong field film flows on the beaker walls it does so with virtually no viscous friction that suggests a thin viscous boundary layer above which a Bernoulli flow may take place. An experimental test of this idea is explored in Sec. VIII.

We here employ the usual notion of a fluid derivative operator,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{v}\cdot\mathrm{grad}), \tag{40}$$

which expresses the time rate of change operator as seen by an observer moving locally with the fluid velocity $\mathbf{v}$. For example, mass conservation reads [15]

$$\frac{d\rho}{dt} = -\,\rho\,\mathrm{div}\,\mathbf{v}. \tag{41}$$

Bernoulli flows are adiabatic, i.e., viscous entropy production is ignored. Conservation of energy then amounts to a local *entropy* conservation law [15],

$$\frac{ds}{dt} = -\,s\,\mathrm{div}\,\mathbf{v}, \tag{42}$$

wherein s is the entropy per unit volume. From Eqs. (40) and (41) it follows that

$$\frac{d}{dt}\left(\frac{s}{\rho}\right) = \frac{1}{\rho^2}\left(\rho\frac{ds}{dt} - s\frac{d\rho}{dt}\right) = 0, \tag{43}$$

which implies a conservation law for the entropy per unit mass,

$$s^* \equiv \frac{s}{\rho} \Rightarrow \frac{ds^*}{dt} = 0. \tag{44}$$

It is here that the enthalpy per unit mass w makes its way into the adiabatic polar dielectric liquid Bernoulli flows in an electric field,

$$\zeta = w - Ts^*,$$

$$dw = Tds^* + \frac{1}{\rho}dP - \frac{1}{4\pi\rho}\mathbf{D}\cdot d\mathbf{E}, \tag{45}$$

wherein Eq. (8) has been invoked. For an adiabatic flow with $ds^*=0$ as in Eq. (44), Eq. (45) implies

$$\rho \text{ grad } w = \text{grad } P - \frac{1}{4\pi}(\mathbf{D}\cdot\text{grad})\mathbf{E}. \tag{46}$$

Thus, the Maxwell pressure tensor force per unit volume [Eq. (32)] in an adiabatic Bernoulli flow reads

$$\mathbf{f} = -\rho \text{ grad } w. \tag{47}$$

The dynamical equation of motions that accounts for momentum conservation then becomes

$$\rho\frac{d\mathbf{v}}{dt} = \mathbf{f} + \rho\mathbf{g},$$

$$\frac{d\mathbf{v}}{dt} = -\text{ grad}(w + \Phi), \tag{48}$$

wherein the gravitational field [Eq. (35)] has been taken into account. Vorticity,

$$\mathbf{\Omega} = \text{curl } \mathbf{v}, \tag{49}$$

makes an appearance in virtue of the acceleration identities,

$$\frac{d\mathbf{v}}{dt} = \frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\text{grad})\mathbf{v},$$

$$\frac{d\mathbf{v}}{dt} = \frac{\partial\mathbf{v}}{\partial t} + \mathbf{\Omega}\times\mathbf{v} + \frac{1}{2}\text{grad}(v^2), \tag{50}$$

which allow us to write Eq. (48) as

$$\frac{\partial\mathbf{v}}{\partial t} + \mathbf{\Omega}\times\mathbf{v} = -\text{ grad}\left(w + \Phi + \frac{1}{2}v^2\right). \tag{51}$$

Employing the curl of Eq. (51) and using Eq. (49) imply the equation of motion for vorticity. It is

$$\frac{\partial\mathbf{\Omega}}{\partial t} + \text{curl}(\mathbf{\Omega}\times\mathbf{v}) = 0,$$

$$\frac{d\mathbf{\Omega}}{dt} = (\mathbf{\Omega}\cdot\text{grad})\mathbf{v} - \mathbf{\Omega}(\text{div } \mathbf{v}). \tag{52}$$

If at a given initial time the Bernoulli flow is irrotational, i.e., $\mathbf{\Omega}=0$, then at all later times in accordance with Eq. (52) the flow will remain irrotational. Equation (51) then reads [15]

$$\mathbf{v} = \text{grad } \varphi,$$

$$\frac{\partial\varphi}{\partial t} + \frac{1}{2}|\text{grad } \varphi|^2 + w + \Phi = 0,$$

$$\frac{\partial\mathbf{v}}{\partial t} = -\text{ grad}\left(w + \Phi + \frac{1}{2}v^2\right). \tag{53}$$

The complete set of equations for a steady state Bernoulli flow in a strong electrostatic field $\mathbf{E}$ and in a uniform gravitational field $\mathbf{g}$ in the negative z direction then follows from Eq. (53) as

$$\text{curl } \mathbf{E} = 0,$$

$$\text{div } \mathbf{D} = \text{div}(\varepsilon\mathbf{E}) = 0,$$

$$w(s^*,P,\mathbf{E}) + \tfrac{1}{2}|\mathbf{v}|^2 + gz = \text{const.} \tag{54}$$

The only difference between the normal steady state Bernoulli fluid flows in Eq. (54) and the usual case for $\mathbf{E}=0$ resides in the electric field contributions to the enthalpy per unit mass w. It is here useful to introduce a new thermodynamic potential per unit mass ϖ obeying

$$w = \varpi + \frac{P}{\rho}, \tag{55}$$

$$d\varpi = Tds^* + \frac{P}{\rho^2}d\rho - \frac{1}{4\pi\rho}\mathbf{D}\cdot d\mathbf{E},$$

$$\varpi(s^*,\rho,E^2) = \varpi_0(s^*,\rho) - \frac{1}{8\pi\rho}\int_0^{E^2}\varepsilon(s^*,\rho,F^2)d(F^2),$$

$$\varpi(s^*,\rho,E^2) = \varpi_0(s^*,\rho) - \frac{\varepsilon(s^*,\rho,0)E^2}{8\pi} + \cdots. \tag{56}$$

For an adiabatic ($ds^*=0$) incompressible ($d\rho=0$) steady state Bernoulli flow, Eqs. (54)–(56) imply the central result of this section,

$$P + \frac{1}{2}\rho v^2 + \rho gz - \frac{\epsilon E^2}{8\pi} = \text{const.} \tag{57}$$

In the case that $\mathbf{E}=0$, the Bernoulli equation [Eq. (41)] indicates that water under the influence of gravity alone flows down a wall and may splash at the bottom. On the other hand, if the electric field on the bottom of the wall is negligible and the electric field on the top of wall is large, then a polar dielectric fluid can crawl up a wall and may splash at the top. Such top splash processes on both of two beakers has been a precursor for building a water bridge.

VIII. CONCLUSIONS

We have discussed the water bridge as a flexible cable made up of pure de-ionized water. The resulting electric fields within the de-ionized water flexible cable maintain a tension that sustains the water against the downward force of gravity. A detailed calculation of the water bridge tension was provided in terms of the Maxwell pressure tensor in a dielectric fluid medium. There has been considerable work done on the microscopic structure of water. The conventional microscopic models [13,14] allow for strongly fluctuating

electric dipole moments but do not include the coupling of these moments to the radiation electromagnetic fields. When model oscillating dipole moments are allowed to radiate [16–20], then it has been shown that ferroelectric domains are formed.

Considering water as an electric ferrofluid subject to high electric fields allows for structures that are more than just a bit unusual. An electric field directed parallel to the water cylinder axis can create a tension as in a stretched rubber band but with different causes. In the case of the rubber band, the tension arises from the high entropy of random knotted polymer chains. In a polar liquid acting as a ferrofluid, the tension arises out of long ordered chains of low entropy aligned coherent dipolar domains.

The resulting tension in the water bridge sustains a siphon between two beakers without the requirement that a new external siphon tube structure made up of other materials be introduced. The Bernoulli flow that can be tested in the laboratory is in the observation of siphon (inverted U-tube) oscillations. Let Λ represents the *effective flow length of the path* from across the risen water surface in one beaker up the beaker wall across the bridge and down the other beaker wall to the other risen water surface. The frequency f of siphon oscillations employing a simple Bernoulli argument is given by

$$f = \frac{1}{\pi}\sqrt{\frac{g}{\Lambda}} \approx \frac{10\ \mathrm{Hz}}{\sqrt{\Lambda/\mathrm{cm}}}. \tag{58}$$

Even more important is the quality factor of the siphon oscillator. If the oscillation lasts a very long time before decaying, as does in a superfluid U-tube oscillation [21], then a strong experimental case can be made for an ideal film flow with very small viscous damping.

APPENDIX: CATENARY BRIDGE

To compute the catenary shape $y(x)$ of a uniform in mass density $\hat{\mu}$ hanging flexible cable, one may invoke the free energy minimum principle. (i) The free energy of the cable is the sum of a gravitational term and a tension term,

$$\mathcal{F}_{\mathrm{cable}} = \hat{\mu} g \int y ds + \tau \int ds,$$

$$\mathcal{F}_{\mathrm{cable}} = \int_0^L F(y',y)dx,$$

$$F(y',y) = (\hat{\mu} g y + \tau)\sqrt{1+y'^2}, \tag{A1}$$

wherein $ds = \sqrt{dx^2+dy^2} = \sqrt{1+y'^2}dx$. The minimum free energy principle then yields the Euler-Lagrange equation,

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \left(\frac{\partial F}{\partial y}\right) \quad \text{with} \quad y(0) = y(L) = 0. \tag{A2}$$

Integrating Eq. (A2) along the string yields

$$\int_0^L \left(\frac{\partial F}{\partial y}\right)dx = \widehat{\mu g} \int ds = Mg,$$

$$\int_0^L \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)dx = 2\tau \sin\,\theta \equiv \tau\left(\frac{y'(L)}{\sqrt{1+y'(L)^2}} - \frac{y'(0)}{\sqrt{1+y'(0)^2}}\right), \tag{A3}$$

wherein $\tan\,\theta = y'(L) = -y'(0)$ has been invoked. The overall mechanical equilibrium [Eq. (29)] follows from Eq. (A3). (ii) The tension at the minimum point is the constant

$$\tau^* = F - y'\frac{\partial F}{\partial y'} = \frac{\hat{\mu} g y + \tau}{\sqrt{1+y'^2}} = \tau \cos\,\theta. \tag{A4}$$

The first-order differential equation [Eq. (A4)] has the solution as given in Eq. (28). The length of the catenary curve is

$$L_s = \int_0^L \sqrt{1+y'^2}dx = 2a\,\sinh\left(\frac{L}{2a}\right) = 2a\,\tan\,\theta, \tag{A5}$$

so that

$$\frac{L_s}{L} = \frac{2\,\tan\,\theta}{\ln[(1+\sin\,\theta)/(1-\sin\,\theta)]}. \tag{A6}$$

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